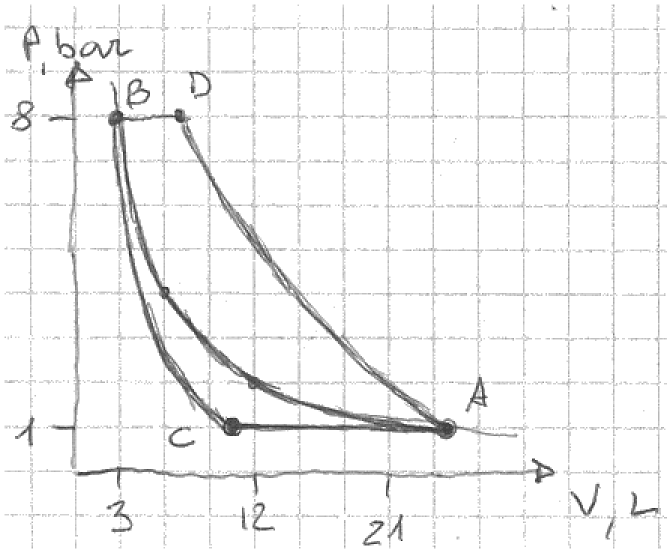
Problem 1 Brayton cycle

Students have made a device capable to operate in a mode that is close to the ideal Brayton cycle. The device can operate in the following modes: 1) reversible adiabatic expansion or compression, 2) reversible isobaric cooling or heating. Through a number of cooling and compression steps, helium is going from the initial state with the pressure of 1 bar and the temperature of 298 K into the final state with the pressure of 8 bar and the temperature of 298 K. (The total number of cooling and compression stages can be from two up to infinity).

1. What is the minimum work that should be done on the gas for this? Compare this value to the work during a reversible isothermal compression.

Solution.
Let's consider the following graphic. The minimum work to cool and compress helium from A (1 bar, 298 K) to B (8 bar, 298 K) is W_{ACB} obtained with an isobaric cooling AC and an adiabatic compression CB.



Points A, B, C, D are characterized by the following data:

A	P = 1 bar	T = 298 K		V_A = 24.78 L
B	P = 8 bar	T = 298 K		V_B = 3.097 L
C	P = 1 bar	$V_C = (P_B/P_C)^{0.6}\ V_B$	$V_C = 8^{0.6}\ 3.097$	V_C = 10.78 L
	T = PV/R	$T = 10^5\ 10.78\ 10^{-3} / 8.31$		T = 129.7 K
D	P = 8 bar	$V_D = (P_A/P_D)^{0.6}\ V_A$	$V_D = (1/8)^{0.6}\ 24.78$	V_D = 7.116 L
	T = PV/R	$T = 8\ 10^5\ 7.116\ 10^{-3} / 8.31$		T = 684.7 K

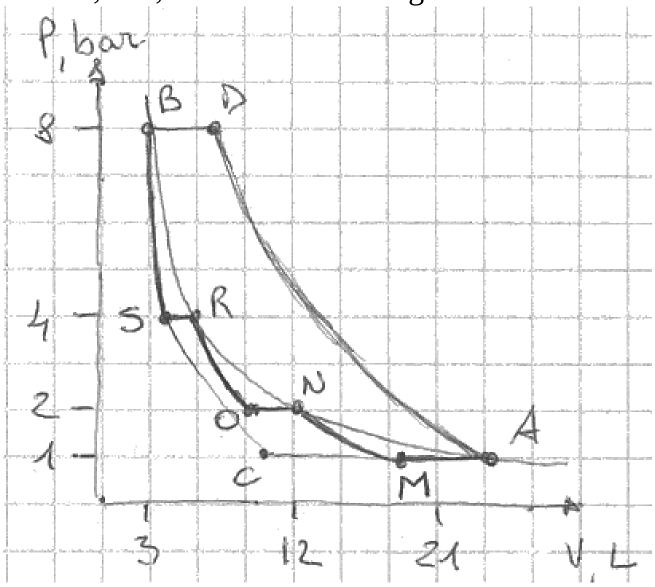
W_{AC} isobaric = $P\Delta V = 1\ 10^5\ (10.78 - 24.78)\ 10^{-3}$ W_{AC} = -1400 J
W_{CB} adiabatic = $- n\ C_v\ \Delta T = - 3/2\ R\ (298 - 129.7)$ W_{CB} = -2099 J
$W_{min} = W_{AC}$ isobaric + W_{CB} adiabatic = -1400 J -2099 J = - 3499 J W_{min} = - 3499 J
W_{AB} isothermic = $P\Delta V = n\ R\ T\ dV/V = RT \ln V_2/V_1$
$W_{AB} = 8.31\ 298 \ln 3.097/24.78$ $W_{AB} = 8.31\ 298 \ln 1/8$ W_{AB} = - 5152 J
The work W_{AB}, done during the isothermal compression, is higher.

2. What is the maximum work that can be done on the gas in this process?

Solution
$W_{max} = W_{ADB} = W_{AD}$ adiabatic + W_{DB} isobaric
W_{AD} adiabatic = $- n\ C_v\ \Delta T = - 3/2\ R\ (684.7 - 298) = - 4823$ J
W_{DB} isobaric = $P\Delta V$ = 8 105 (3.097 - 7.116) 10-3 = - 3215 J
W_{max} = - 4823 - 3215 = - 8038 J

3. Let the process be accomplished in three steps. At each step helium is first cooled and then compressed. At the end of each step the pressure increases twice and the temperature returns to the value of 298 K. What is the total heat removed from the gas by a Peltier element?

Solution
Let's consider the following graphic.
AM-MN, NO-OR, RS-SB are the three steps of isobaric and adiabatic transformations
Heat is removed only on AM, NO, RS isobaric coolings.



Points N, M, R, O are characterized by the following data:

N	P = 2 bar	T = 298 K	$PV = P_BV_B$ $\quad V = P_B/P\ V_B$ $\quad V = 8/2\ 3.097$	$V_N = 12.39$ L
M	P = 1 bar	$V = (P_N/P)^{0.6}\ V_N$	$V = 2^{0.6}\ 12.39$	$V_M = 18.78$ L
	T = PV/R	$T = 10^5\ 18.78\ 10^{-3} / 8.31$		T = 225.9 K
R	P = 4 bar	T = 298 K	$PV = P_BV_B$ $\quad V = P_B/P\ V_B$ $\quad V = 8/4\ 3.097$	$V_R = 6.194$ L
O	P = 2 bar	$V = (P_P/P)^{0.6}\ V_P$	$V = 2^{0.6}\ 6.194$	$V_O = 9.388$ L
	T = PV/R	$T = 2\ 10^5\ 9.388\ 10^{-3} / 8.31$		T = 225.9 K

$Q_{AM} = n\ Cp\ \Delta T$ $\qquad Q_{AM} = 5/2\ R\ (225.9 – 298)$ $\qquad Q_{AM} = 1499$ J
$Q_{NO} = n\ Cp\ \Delta T$ $\qquad Q_{NO} = 5/2\ R\ (225.9 – 298)$ $\qquad Q_{NO} = 1499$ J
$Q_{total} = Q_{AM} + Q_{NO} + Q_{PR} = 3\ \ 1499$ $\qquad Q_{total} = 4497$ J
(heat removed from the gas during the three steps tranformation)

Once the gas is compressed, it is returned to the initial state (1 bar and 298 K) in two stages (heating and expansion).
4. What is the range of possible values of the formal efficiency η for the resulting cycle? η is the ratio of the useful work done by the gas to the amount of heat given to the gas during the heating stage.

Solution
$\eta = W/Q_{BD}$ $\qquad \eta_{max} = W_{max}/Q_{BD}$ $\qquad \eta_{min} = W_{min}/Q_{BD}$
$W_{max} = W_{BDACB}$ $\qquad W_{min} = W_{BDAB}$
$W_{max} = W_{BD} + W_{DA} + W_{AC} + W_{CB}$ $\qquad W_{max} = 3215 + 4823 -1400 -2099 = 8038 - 3499 = 4539$ J
$W_{min} = W_{BD} + W_{DA} + W_{AB}$ $\qquad W_{min} = 3215 + 4823 - 5152 = 8038 - 5152 = 2886$ J
$Q_{BD} = n\ Cp\ \Delta T = 5/2\ R\ (684.7 - 298)$ $\qquad Q_{BD} = 8038$ J
$\eta_{max} = W_{max}/Q_{BD}$ $\qquad \eta_{max} = 4539/8038 = 0.565$
$\eta_{min} = W_{min}/Q_{BD}$ $\qquad \eta_{min} = 2886/8038 = 0.359$

5. In one of the experiments, the gas has been compressed from 1 bar and 298 K to 8 bar and 298 K in several steps (like in question 3). At the end of each step the pressure is increased by x times and the temperature returns to 298 K. Then helium has been returned to the initial state in two stages – heating and expansion. Theoretical value of η for this cycle is 0.379. How many steps were used?

Solution
In this cycle, named X, we have $\eta = 0.379$ $\eta = W_{done}/Q_{absorbed}$
$W_{done} = \eta\ Q_{absorbed}$ $W_{done} = 0.379\ 8038 = 3046.4$ J
$W_{done} = W_{BD} + W_{DA} + Wx$ $Wx = W_{done} - W_{BD} - W_{DA}$ $Wx = 3046.4 - 8038 = -4992.4$ J
Wx = recharge work in cycle X
The equation that links the increment of pression x to the number of cycles n is:
$x^n = 8$ $x = 8^{1/n}$
In question 3 we had three cycles so n = 3 and x = 2. In fact we had
$P_A = x^0 = 2^0 = 1$ bar $P_N = x^1 = 2^1 = 2$ bar $P_R = x^2 = 2^2 = 4$ bar $P_B = x^3 = 2^3 = 8$ bar
$W_{AMN} = Q_{AM}$
Points N, M, O, R are characterized by the following data:

M	$P = x^0$ bar	$V_M = (P_N/P_M)^{0.6}\ V_N$	$V_M = (x^1/x^0)^{0.6}\ 8/x\ 3.097$ L	$V_M = x^{0.6}\ 8/x\ 3.097$ L
	$T = PV/R = x^0\ x^{0.6}\ 8/x\ 3.097\ 10^2/R$			$T_M = x^{0.6}\ 8/x\ 3.097\ 10^2/R$
N	$P = x^1$ bar	$V_N = P_B/P_N\ V_B$	$V_N = 8/x\ 3.097$ L	$T_N = T_A = 298$ K
O	$P = x^1$ bar	$V_O = (P_R/P_O)^{0.6}\ V_R$	$V_O = (x^2/x^1)^{0.6}\ 8/x^2\ 3.097$ L	
	$V_O = x^{0.6}\ 8/x^2\ 3.097$	$V_O = V_M/x$	$T = PV/R = x\ x^{0.6}\ 8/x^2\ 3.097\ 10^2/R$	
	$T = x^{0.6}\ 8/x\ 3.097\ 10^2/R$		$T_O = T_M$ M, O, S are on the same isotherm	
R	$P = x^2$ bar	$V_R = P_B/P_R\ V_B$	$V_R = 8/x^2\ 3.097$ L	

$Q_{AM} = 1\ \text{mol}\ Cp\ \Delta T = 5/2\ R\ (T_M - T_A) = 5/2\ R\ (x^{0.6}\ 8/x\ 3.097\ 10^2/R - 298)$ $Q_{AM} = 6194\ (x^{-0.4} - 1)$ J
where $x = 8^{1/n}$ then $Q_{AM} = 6194\ (8^{-0.4})^{1/n} - 1)$
remember that $T_M = T_O = T_S$ ΔT is always the same, then $Q_{AM} = Q_{NO} = Q_{RS} = ...$ etc
$W_{total\ AMNORS...B} = Wx = n\ Q_{AM}$
The equation becomes $n\ Q_{AM} = -4992.4$ J $n\ 6194\ [(8^{-0.4})^{1/n} - 1] = -4992.4$
$n\ [(8^{-0.4})^{1/n} - 1] = -0.806$
Solving this equation we obtain n = 13
The number of cycles that produces η = 0.379 is 13.

In fact, Peltier elements also consume electric energy during the cooling stage. Assume that they consume as much energy as is removed from the gas.

6. What is the maximum possible efficiency of the considered cycle, taking into account energy consumption during cooling?

Solution
Useful work = $W_{BDA} = 8038$ J
Minimum work to close the cycle = $W_{ACB} = -3499$ J
Energy requested to extract heat = $Q_{ACB} = W_{ACB} = -3499$ J
Heat given to the gas during isobaric heating BD = $Q_{BD} = W_{BDA} = 8038$ J
Maximum formal efficiency of the cycle = $\eta = (W_{BDA} + W_{ACB} + Q_{ACB}) / Q_{BD}$
$\eta = (W_{BDA} + 2\ W_{ACB}) / W_{BDA}$ $\eta = (8038 - 2\ \ 3499) / 8038$ $\eta = 0{,}129$

Solution proposed by
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